Chapter 10

Case studies: multiple constraints and compound objectives

10.1 Introduction and synopsis

These case studies illustrate how the techniques described in the previous chapter really work. Two were sketched out there: the light, *stiff*, *strong* beam, and the *light*, *cheap*, stiff beam. Here we develop four more. The first pair illustrate multiple constraints; here the *active constraint* method is used. The second pair illustrate compound objectives; here a *value function* containing an exchange constant, $E^{\$}$, is formulated. The examples are deliberately simplified to avoid clouding the illustration with unnecessary detail. The simplification is not nearly as critical as it may at first appear: the choice of material is determined primarily by the physical principles of the problem, not by details of geometry. The principles remain the same when much of the detail is removed so that the selection is largely independent of these.

Further case studies can be found in the sources listed under Further reading.

10.2 Multiple constraints – con-rods for high-performance engines

A connecting rod in a high performance engine, compressor or pump is a critical component: if it fails, catastrophe follows. Yet — to minimize inertial forces and bearing loads — it must weigh as little as possible, implying the use of light, strong materials, stressed near their limits. When cost, not performance, is the design goal, con-rods are frequently made of cast iron, because it is so cheap. But what are the best materials for con-rods when performance is the objective?

The model

Table 10.1 summarizes the design requirements for a connecting rod of minimum weight with two constraints: that it must carry a peak load F without failing either by fatigue or by buckling elastically. For simplicity, we assume that the shaft has a rectangular section $A = bw$ (Figure 10.1).

The objective function is an equation for the mass which we approximate as

$$m = \beta A L \rho \qquad (10.1)$$

where L is the length of the con-rod and ρ the density of the material of which it is made, A the cross-section of the shaft and β a constant multiplier to allow for the mass of the bearing housings.

Table 10.1 The design requirements: connecting rods

Function	Connecting rod for reciprocating engine or pump
Objective	Minimize mass
Constraints	(a) Must not fail by high-cycle fatigue, or (b) Must not fail by elastic buckling (c) Stroke, and thus con-rod length L, specified

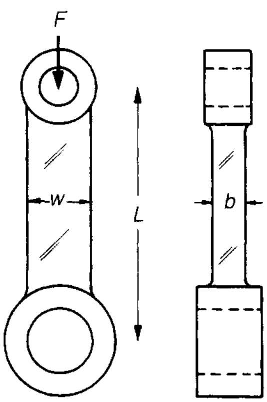

Fig. 10.1 A connecting rod. The rod must not buckle, fail by fatigue or by fast fracture (an example of multiple constraints). The objective is to minimize mass.

The fatigue constraint requires that

$$\frac{F}{A} \leq \sigma_e \tag{10.2}$$

where σ_e is the endurance limit of the material of which the con-rod is made. (Here, and elsewhere, we omit the safety factor which would normally enter an equation of this sort, since it does not influence the selection.) Using equation (10.2) to eliminate A in equation (10.1) gives the mass of a con-rod which will just meet the fatigue constraint:

$$m_1 = \beta F L \left(\frac{\rho}{\sigma_e}\right) \tag{10.3}$$

containing the material index

$$\boxed{M_1 = \frac{\sigma_e}{\rho}} \tag{10.4}$$

The buckling constraint requires that the peak compressive load F does not exceed the Euler buckling load:

$$F \leq \frac{\pi^2 E I}{L^2} \tag{10.5}$$

with $I = b^3 w/12$. Writing $b = \alpha w$, where α is a dimensionless 'shape-constant' characterizing the proportions of the cross-section, and eliminating A from equation (10.1) gives a second equation for the mass

$$m_2 = \beta \left(\frac{12F}{\alpha \pi^2}\right)^{1/2} L^2 \left(\frac{\rho}{E^{1/2}}\right) \tag{10.6}$$

containing the material index (the quantity we wish to maximize to avoid buckling):

$$M_2 = \frac{E^{1/2}}{\rho} \tag{10.7}$$

The con-rod, to be safe, must meet both constraints. For a given stroke, and thus length, L, the active constraint is the one leading to the largest value of the mass, m. Figure 10.2 shows the way in which m varies with L (a sketch of equations (10.3) and (10.6)), for a single material: short con-rods are liable to fatigue failure, long ones are prone to buckle.

The selection: analytical method

Consider first the selection of a material for the con-rod from among those listed in Table 10.2. The specifications are

$$L = 150\,\text{mm} \qquad F = 50\,\text{kN} \qquad \alpha = 0.5 \qquad \beta = 1$$

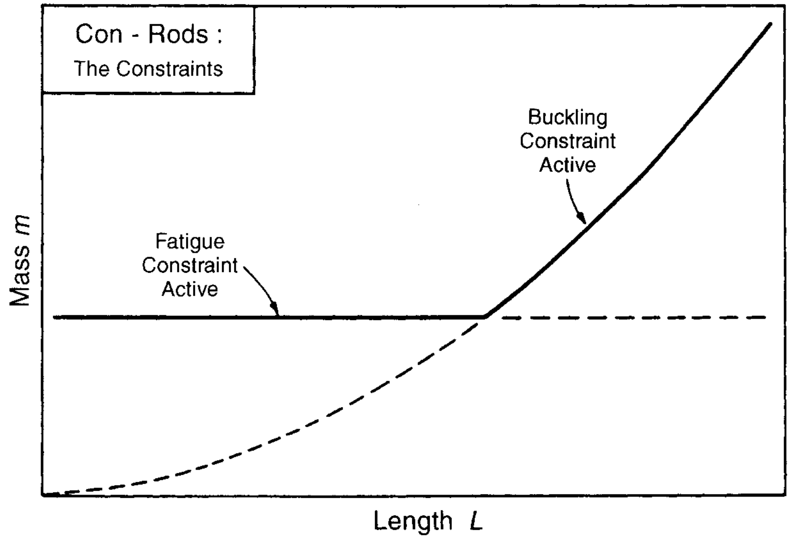

Fig. 10.2 The equations for the mass m of the con-rod are shown schematically as a function of L.

Table 10.2 Selection of a material for the con-rod

Material	ρ kg/m^3	E GPa	σ_e MPa	m_1 kg	m_2 kg	$\tilde{m}$ kg
Nodular cast iron	7150	178	250	**0.21**	0.13	0.21
HSLA steel 4140 (o.q. T-315)	7850	210	590	0.1	**0.13**	0.13
Al 539.0 casting alloy	2700	70	75	**0.27**	0.08	0.27
Duralcan Al-SiC(p) composite	2880	110	230	**0.09**	0.07	**0.09**
Ti-6-4	4400	115	530	0.06	**0.1**	0.1

The table lists the mass m_1 of a rod which will just meet the fatigue constraint, and the mass m_2 which will just meet that on buckling (equations (10.3) and (10.6)). For three of the materials the active constraint is that of fatigue; for two it is that of buckling. The quantity $\tilde{m}$ in the last column of the table is the larger of m_1 and m_2 for each material; it is the lowest mass which meets both constraints. The material offering the lightest rod is that with the smallest value of $\tilde{m}$. Here it is the metal-matrix composite Duralcan 6061–20% SiC(p). The titanium alloy is a close second. Both weigh about half as much as a cast-iron rod.

The selection: graphical method

The mass of the rod which will survive both fatigue and buckling is the larger of the two masses m_1 and m_2 (equations (10.3) and (10.6)). Setting them equal gives the equation of the coupling line:

$$M_2 = \left[\left(\frac{12L^2}{\pi^2 \alpha F} \right)^{1/2} \right] M_1 \tag{10.8}$$

The quantity in square brackets is the coupling constant: it contains the quantity F/L^2 — the 'structural loading coefficient' of Chapter 5.

Materials with the optimum combination of M_1 and M_2 are identified by creating a chart with these indices as axes. Figure 10.3 illustrates this, using a database of light alloys. Coupling lines for two values of F/L^2 are plotted on it, taking $\alpha = 0.5$. Two extreme selections are shown, one isolating the best subset when the structural loading coefficient F/L^2 is high, the other when it is low. For the high value ($F/L^2 = 0.5\,\text{MPa}$), the best materials are high-strength Mg-alloys, followed by high-strength Ti-alloys. For the low value ($F/L^2 = 0.05\,\text{MPa}$), beryllium alloys are the optimum choice. Table 10.3 lists the conclusions.

Postscript

Con-rods have been made from all the materials in the table: aluminium and magnesium in family cars, titanium and (rarely) beryllium in racing engines. Had we included CFRP in the selection, we would have found that it, too, performs well by the criteria we have used. This conclusion has been reached by others, who have tried to do something about it: at least three designs of CFRP con-rods have been prototyped. It is not easy to design a CFRP con-rod. It is essential to use continuous fibres, which must be wound in such a way as to create both the shaft and the bearing housings; and the shaft must have a high proportion of fibres which lie parallel to the direction in which F acts. You might, as a challenge, devise how you would do it.

Table 10.3 Materials for high-performance con-rods

Material	*Comment*
Magnesium alloys	ZK 60 and related alloys offer good all-round performance.
Titanium alloys	Ti-6-4 is the best choice for high F/L^2.
Beryllium alloys	The ultimate choice when F/L^2 is small. Difficult to process.
Aluminium alloys	Cheaper than titanium or magnesium, but lower performance.

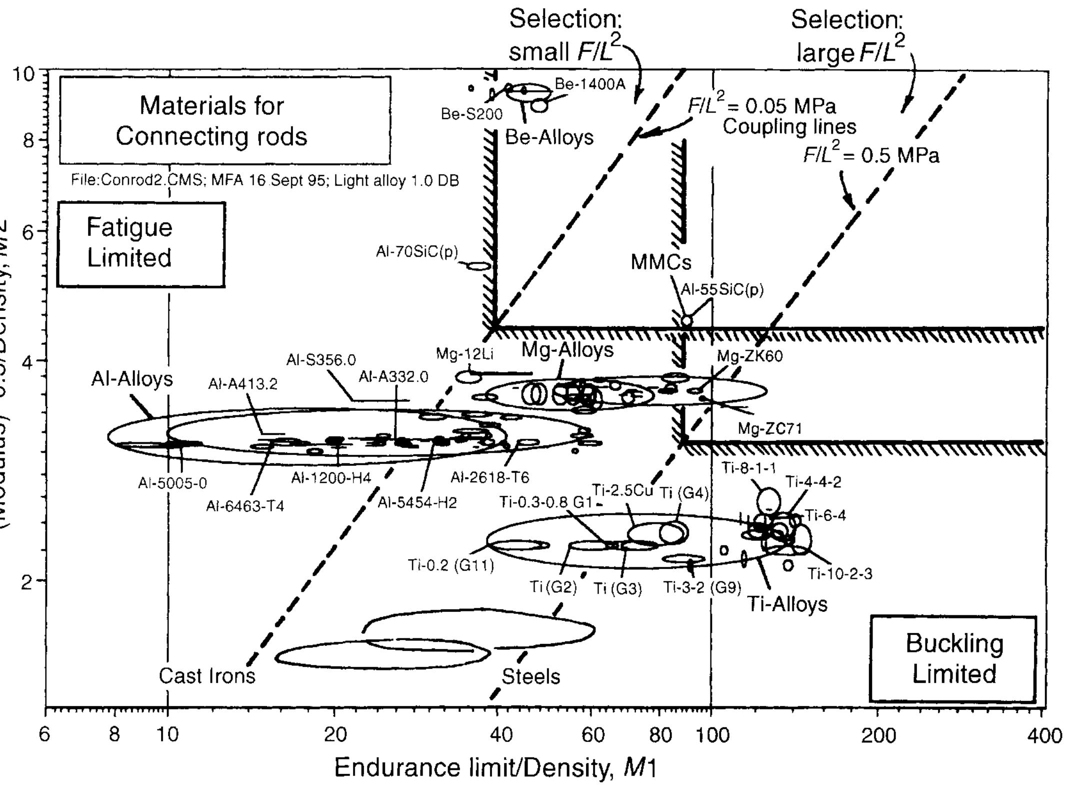

Fig. 10.3 Over-constrained design leads to two or more performance indices linked by coupling equations. The diagonal broken lines show the coupling equations for two values of the coupling constant, determined by the 'structural loading coefficient' F/L^2. The two selection lines must intersect on the appropriate coupling line giving the box-shaped search areas. (Figure created using *CMS* (1995) software.)

Related case studies

Case Study 10.3: Multiple constraints — windings for high field magnets

10.3 Multiple constraints – windings for high field magnets

Physicists, for reasons of their own, like to see what happens to things in high magnetic fields. 'High' means 50 tesla or more. The only way to get such fields is the old-fashioned one: dump a huge current through a wire-wound coil; neither permanent magnets (practical limit: 1.5 T), nor super-conducting coils (present limit: 25 T) can achieve such high fields. The current generates a field-pulse which lasts as long as the current flows. The upper limits on the field and its duration are set by the material of the coil itself: if the field is too high, the coil blows itself apart; if too long, it melts. So choosing the right material for the coil is critical. What should it be? The answer depends on the pulse length.

Table 10.4 Duration and strengths of pulsed fields

Classification	*Duration*	*Field strength*
Continuous	1 s–∞	<30 T
Long	100 ms–1 s	30–60 T
Standard	10–100 ms	40–70 T
Short	10–1000 μs	70–80 T
Ultra-short	0.1–10 μs	>100 T

Pulsed fields are classified according to their duration and strength as in Table 10.4.

The model

The magnet is shown, very schematically, in Figure 10.4. The coils are designed to survive the pulse, although not all do. The requirements for survival are summarized in Table 10.5. There is one objective — to maximize the field — with two constraints which derive from the requirement of survivability for a given pulse length.

Consider first destruction by magnetic loading. The field, B (units: weber/m^2), in a long solenoid like that of Figure 10.4 is:

$$B = \frac{\mu_o N i}{\ell} \lambda_f F(\alpha, \beta) \qquad (10.9)$$

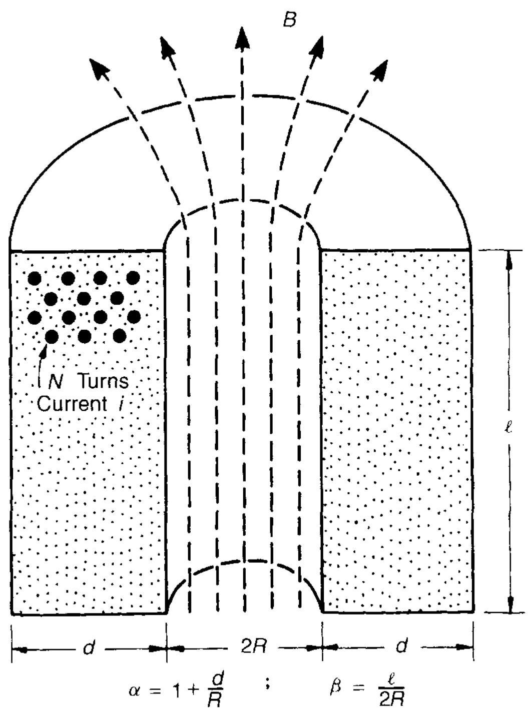

Fig. 10.4 Windings for high-powered magnets. There are two constraints: the magnet must not overheat; and it must not fail under the radial magnetic forces.

Table 10.5 The design requirements: high field magnet

Function	Magnet windings
Objective	Maximize magnetic field
Constraints	(a) No mechanical failure (b) Temperature rise <150°C (c) Radius R and length ℓ of coil specified

where μ_o is the permeability of air ($4\pi \times 10^{-7}$ Wb/Am), N is the number of turns, i is the current, ℓ is the length of the coil, λ_f is the filling-factor which accounts for the thickness of insulation (λ_f = cross-section of conductor/cross section of coil), and $F(\alpha, \beta)$ is a geometric constant (the 'shape factor') which depends on the proportions of the magnet (defined on Figure 10.4), the value of which need not concern us. The field creates a force on the current-carrying coil. It acts radially outwards, rather like the pressure in a pressure vessel, with a magnitude

$$p = \frac{B^2}{2\mu_o F(\alpha, \beta)} \tag{10.10}$$

though it is actually a body force, not a surface force. The pressure generates a stress σ in the windings and their casing

$$\sigma = \frac{pR}{d} = \frac{B^2}{2\mu_o F(\alpha, \beta)} \frac{R}{d} \tag{10.11}$$

This must not exceed the yield strength σ_y of the windings, giving the first limit on B:

$$B_1 \leq \left(\frac{2\mu_o d \sigma_y F(\alpha, \beta)}{R} \right)^{1/2} \tag{10.12}$$

The field is maximized by maximizing

$$\boxed{M_1 = \sigma_y} \tag{10.13}$$

One could have guessed this: the best material to carry a stress σ is that with the largest yield strength σ_y.

Now consider destruction by overheating. High-powered magnets are initially cooled in liquid nitrogen to -196°C in order to reduce the resistance of the windings; if the windings warm above room temperature, the resistance, R_e, in general, becomes too large. The entire energy of the pulse, $\int i^2 R_e \, dt \approx i^2 \overline{R}_e t_p$ is converted into heat (here $\overline{R}_e$ is the average of the resistance over the heating cycle and t_p is the length of the pulse); and since there is insufficient time for the heat to be conducted away, this energy causes the temperature of the coil to rise by ΔT, where

$$\Delta T = \frac{B^2}{\mu_o^2} \frac{\rho_e t_p}{d^2 C_p \rho} \tag{10.14}$$

Here ρ_e is the resistivity of the material, C_p its specific heat (J/kg K) and ρ its density. The resistance of the coil, R_e, is related to the resistivity of the material of the windings by

$$R_e = \frac{4\ell \rho_e}{\pi d^2}$$

where d is the diameter of the conducting wire. If the upper limit for the temperature is 200 K, $\Delta T_{max} \leq 100$ K, giving the second limit on B:

$$B_2 \leq \left(\frac{\mu_o^2 d^2 C_p \rho \lambda_f \Delta T_{max}}{t_p \rho_e} \right)^{1/2} F(\alpha, \beta) \tag{10.15}$$

The field is maximized by maximizing

$$\boxed{M_2 = \frac{C_p \rho}{\rho_e}} \tag{10.16}$$

The two equations for B are sketched, as a function of pulse-time, t_p, in Figure 10.5. For short pulses, the strength constraint is active; for long ones, the heating constraint is dominant.

The selection: analytical method

Table 10.6 lists material properties for three alternative windings. The sixth column gives the strength-limited field strength, B_1; the seventh column, the heat-limited field B_2 evaluated for the following values of the design requirements:

$$t_p = 10\,\text{ms} \qquad \lambda_f = 0.5 \qquad \Delta T_{max} = 100\,\text{K}$$
$$F(\alpha, \beta) = 1 \qquad R = 0.05\,\text{m} \qquad d = 0.1\,\text{m}$$

Strength is the active constraint for the copper-based alloys; heating for the steels. The last column lists the limiting field $\tilde{B}$ for the active constraint. The Cu–Nb composites offer the largest $\tilde{B}$.

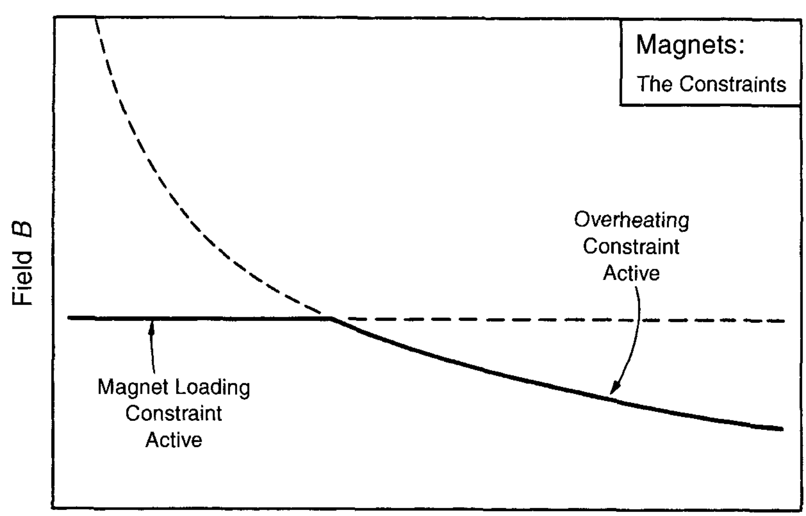

Fig. 10.5 The two equations for B are sketched, indicating the active constraint.

Table 10.6 Selection of a material for a high field magnet, pulse length 10 ms

Material	ρ *Mg/m*^{3}	σ_y *MPa*	C_p *J/kg K*	ρ_e $10^{-8}\,\Omega\,m$	B_1 *Wb/m*^{2}	B_2 *Wb/m*^{2}	$\tilde{B}$ *Wb/m*^{2}
High-conductivity copper	8.94	250	385	1.7	35	**113**	35
Cu–15% Nb composite	8.90	780	368	2.4	62	92	**62**
HSLA steel	7.85	1600	450	25	**89**	30	30

The selection: graphical method

The cross-over lies along the line where equations (10.12) and (10.15) are equal, giving the coupling the line

$$M_1 = \left[\frac{\mu_o R d \lambda_f F(\alpha, \beta)\Delta T_{\max}}{2t_p}\right] M_2 \qquad (10.17)$$

The quantity in square brackets is the coupling constant; it depends on the pulse length, t_p.

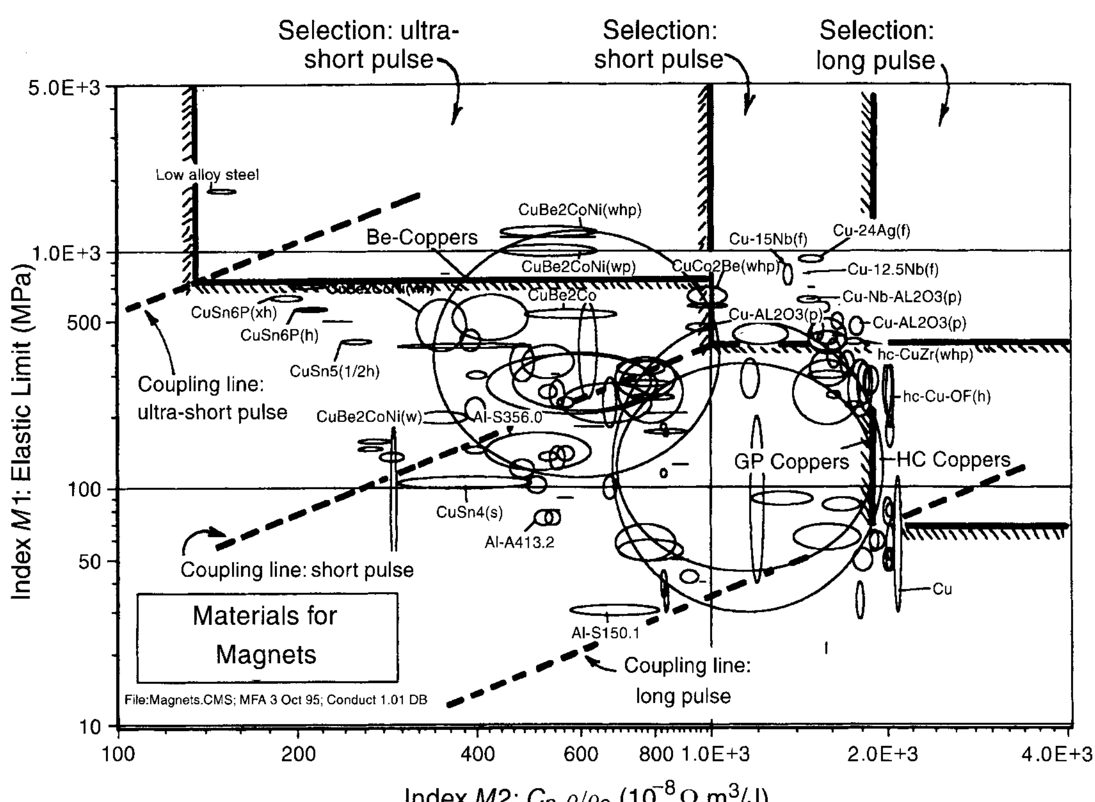

Fig. 10.6 Materials for windings for high-powered magnets, showing the selection for long pulse applications, and for short pulse ultra-high field applications. (Figure created using *CMS* (1995) software.)

Table 10.7 Materials for high field magnet windings

Material	*Comment*
Continuous and long pulse	
High conductivity coppers Pure silver	Best choice for low field, long pulse magnets (heat-limited).
Short pulse	
Copper–AL_2O_3 composites (Glidcop) H–C copper cadmium alloys H–C copper zirconium alloys H–C copper chromium alloys Drawn copper-niobium composites	Best choice for high field, short pulse magnets (heat and strength limited).
Ultra short pulse, ultra high field	
Copper–beryllium–cobalt alloys High-strength, low-alloy steels	Best choice for high field, short pulse magnets (strength-limited).

The selection is illustrated in Figure 10.6. Here we have used a database of *conductors*: it is an example of sector-specific database (one containing materials and data relevant to a specific industrial sector, rather than one that is material class-specific). The axes are the two indices M_1 and M_2. Three selections are shown, one for very short-pulse magnets, the other for long pulses. Each selection box is a contour of constant field, B; its corner lies on the coupling line for the appropriate pulse duration. The best choice, for a given pulse length, is that contained in the box which lies farthest up its coupling line. The results are summarized in Table 10.7.

Postscript

The case study, as developed here, is an oversimplification. Magnet design, today, is very sophisticated, involving nested sets of electro and super-conducting magnets (up to 9 deep), with geometry the most important variable. But the selection scheme for coil materials has validity: when pulses are long, resistivity is the primary consideration; when they are very short, it is strength, and the best choice for each is that developed here. Similar considerations enter the selection of materials for very high-speed motors, for bus-bars and for relays.

Further reading

Herlach, F. (1988) The technology of pulsed high-field magnets, *IEEE Transactions on Magnetics*, **24**, 1049.
Wood, J.T., Embury, J.D. and Ashby, M.F. (1995) An approach to material selection for high field magnet design, submitted to *Acta Metal. et Mater.* **43**, 212.

Related case studies

Case Study 10.2: Multiple constraints — con-rods

10.4 Compound objectives — materials for insulation

The objective in insulating a refrigerator (of which that sketched in Figure 10.7 is one class — there are many others) is to minimize the energy lost from it, and thus the running cost. But the insulation

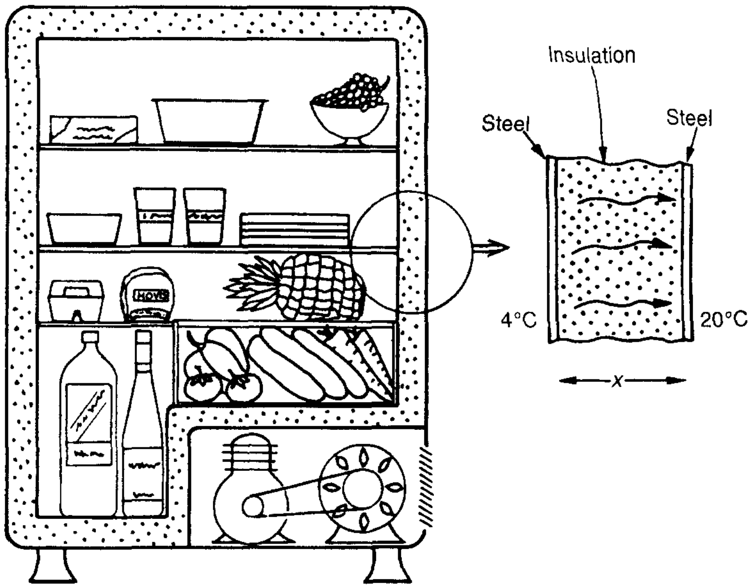

Fig. 10.7 Insulation for refrigerators. The objectives are to minimize heat loss from the interior and to minimize the cost of the insulation itself.

itself has a capital cost associated with it. The most economical choice of material for insulation is that which minimizes the total. There is at least one constraint: an upper limit on the thickness $x_{\max}$ of the insulation (Table 10.8).

The model

The first objective is to minimize the cost of the insulation. This cost, per unit area of wall, is

$$C = x_{\max} \rho C_m \tag{10.18}$$

Here C_m is the cost/kg of the insulation and ρ is its density.

The second objective is to minimize the energy loss. The heat flux per unit area of wall, Q (W/m^2), assuming steady-state heat flow, is

$$Q = -\lambda \frac{dT}{dx} = \frac{\lambda \, \Delta T}{x_{\max}}$$

where λ (W/m K) is its thermal conductivity and ΔT is the temperature difference between the inside and the outside of the insulation layer. If the refrigerator runs continuously, the energy consumed

Table 10.8 Design requirement for refrigerator insulation

Function	Thermal insulation
Objectives	(a) Minimize insulation cost and (b) Minimize energy loss, appropriately coupled
Constraint	Thickness $\leq x_{\max}$

in time t(s) is

$$H = Qt\,(\text{J/m}^2) \tag{10.19}$$

We identify t with the design life of the refrigerator.

To minimize both objectives in a properly couple way we create a value-function, V,

$$V = -C + E^{\$}H$$

with C given by equation (10.18) and H by (10.19). It contains the exchange constant, $E^{\$}$, relating energy to cost. It can vary widely (Table 9.5). If grid-electricity is available, $E^{\$}$ is low. But in remote areas (requiring power-pack generation), in aircraft (supplementary turbine generator) or in space (solar panels), it can be far higher. (The exchange constant relating value to cost is -1, giving the negative sign.) Inserting equations (10.18) and (10.19) gives

$$V = -x_{\max}[\rho C_m] + E^{\$}\left(\frac{t\Delta T}{x_{\max}}\right)[\lambda] \tag{10.20}$$

Here the material properties are enclosed in square brackets; everything outside these brackets is fixed by the design.

The selection: analytical method

Take the example

$$X_{\max} = 20\,\text{mm} \qquad \Delta T = 20\,\text{C} \qquad t = 1\text{ year} = 31.5 \times 10^6\,\text{s}$$
$$E^{\$} = -0.02\ \$/\text{MJ (grid electricity)}$$

giving, for four candidate foams listed in Table 10.9, the values of V shown in the last column.

The polystyrene foam is the cheapest to buy, but the phenolic has the largest (least negative) value of V. It is the best choice.

The selection: graphical method

Define the indices

$$M_1 = \rho C_m \text{ and } M_2 = \lambda$$

We rewrite equation (10.18) in the form:

$$\tilde{V} = \frac{V}{x_{\max}} = -M_1 + E^{\$}\left[\frac{\Delta T}{x_{\max}^2}t\right]M_2 \tag{10.21}$$

Table 10.9 Value function, V, for thermal insulation

Material	ρ *kg/m³*	λ *W/m K*	C_m *\$/kg*	σ_y *MPa*	C *\$/m²*	V $E^{\$} = -0.02$ *\$/MJ*
Polystyrene foam	30	0.034	2.0	0.2	**1.2**	−22.6
Phenolic foam	35	0.025	4.0	0.2	2.8	**−18.6**
Polymethacrylimide foam	50	0.030	27	0.8	27	−45.9
Polyethersulphone foam	90	0.038	18	0.8	32	−56.0

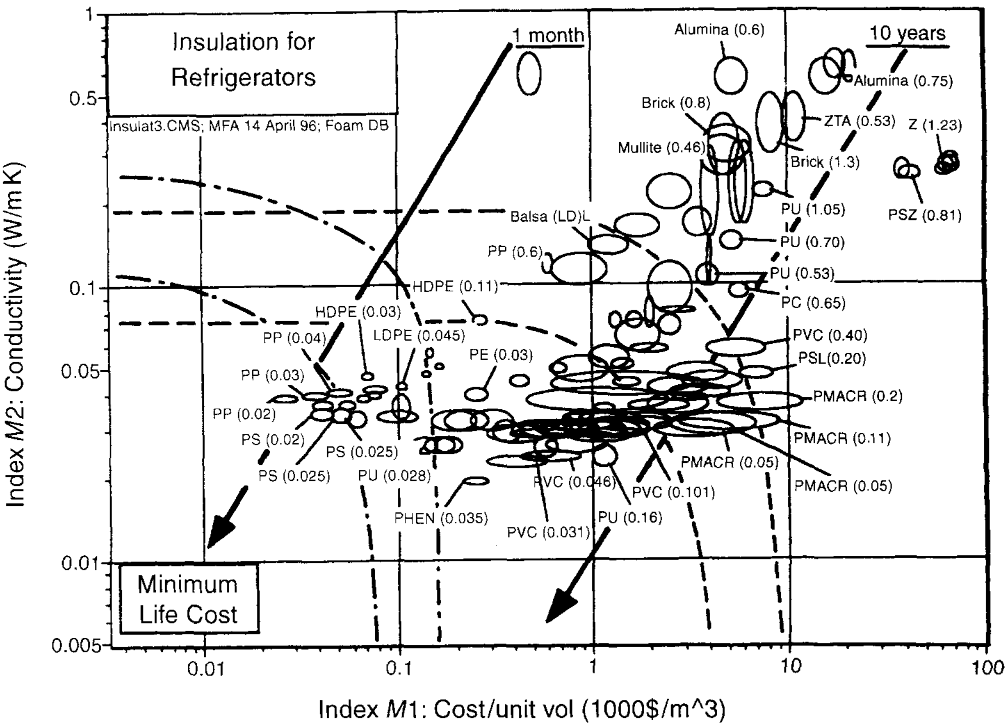

Fig. 10.8 Selection of insulating materials for refrigerators with different design lives. (Figure created using CMS (1997) software.)

Everything in the equation is specified except the material groups M_1 and M_2. We seek materials which maximize $\tilde{V}$. Figure 10.8 shows M_1 plotted against M_2. Contours of constant $\tilde{V}$ appear as curved lines; the value of $\tilde{V}$ increases towards the bottom right. Two sets of contours are shown, one for long-term insulation with a design life, t, of 10 years, the other for short-term refrigeration, with a value of t of 1 month. To plot these, we need a value for the term in square brackets. It has been evaluated using $\Delta T = 20°C$, $x_{max} = 20\,mm$, and $E^{\$} = -0.02$ \$/MJ, as before. For the shorter design life, the cost of the insulation dominates the value function; then the best choices are simply the cheapest ones: the low density expanded polystyrene EPS (0.03) for instance*. But for the longer design life, the second term on the right of equation (10.21) becomes dominant, and the choice of material changes; the contours shown for $t = 10$ years suggest that low-density phenolics might be a good selection, because their conductivity is lower than that of the polystyrenes. Table 10.10 summarizes the selection.

Postscript

In many insulation applications the foam is bonded to the inner and outer walls of the refrigerator to give stiffness: it performs a mechanical as well as a thermal function. Then the strength, σ_y, may also be relevant. The table includes two high-strength foams.

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* On Figure 10.6 the letters identify the material; the number in brackets gives the density in Mg/m^3. Thus PS(0.03) means 'a polystyrene foam with a density of $0.03\,Mg/m^3$'.

Table 10.10 Materials for refrigerator insulation

Material	*Comment*
Short design life (t_ℓ = 1 month)	
Polystyrene (PS) foams, e.g. PS(0.02) or PS(0.025)	Cost of insulation dominates the value function; polystyrene and polypropylene foams are the best choice because they are the cheapest.
Polypropylene (PP) foams, e.g. PP(0.02) or PP(0.03)	
Long design life (t_ℓ = 10 years)	
Phenolic (PHEN) foams, e.g. PHEN(0.035)	Heat conduction is important in the value function. The more expensive phenolics minimize the value function and are the best choice.
Polyurethane (PU) foams, e.g. PU(0.028)	
Polystyrene (PS) foams, e.g. PS(0.02) or PS(0.025)	

Of the two, the polymethacrylimide foam gives the largest (least negative) value of V.

Related case studies

Case Study 10.5: Compound objectives — disposable coffee cups

10.5 Compound objectives — disposable coffee cups

It is increasingly recognized that the use of materials in engineering carries environmental penalties: pollution of water and air, solid waste, consumption of non-renewable resources and more (collectively called *eco-damage*). One response is to adopt, as a design objective, the minimization of this damage.

Consider, as an example, the replacement of an existing disposable cup (Figure 10.9) by one which is more environmentally benign. The environmental impact it causes is difficult to quantify. One component of impact relates to the *energy content* of the material: many aspects of impact (CO_2 emissions, air-borne particulates) are proportional to this. And energy content *can* be quantified, at least approximately. We shall use it as a measure of environmental impact, to illustrate how it can be balanced against cost.

Disposable cups are not, at present, recycled, so the energy and material they contain are irretrievably lost when they are discarded. To minimize the eco-impact (measured now by energy content), we seek the design which incorporates the least energy to start with. But disposable cups must also be cheap. So we find two conflicting objectives: the environmental goal of minimizing energy content, and the economic one of minimizing cost. There are constraints which must be met: the cup must be sufficiently stiff that it can be picked up without ovalizing severely, and it would be desirable, too, that it also insulates (Table 10.11).

We first write a value function for the cup:

$$V = -C + E^{\$}qm \qquad (10.22)$$

Here C is the cost of the cup, m is its mass and q the energy content per unit mass of the material of which it is made. The quantity $E^{\$}$ is the exchange constant: the value associated with one unit

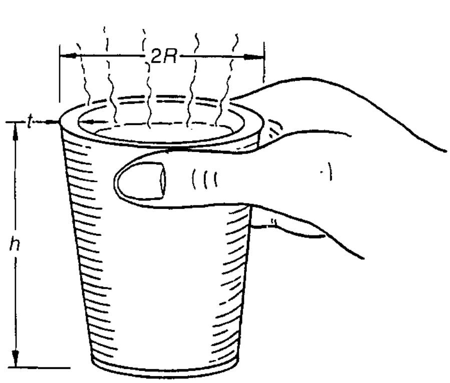

Fig. 10.9 A disposable hot-drink cup. It must be cheap, stiff and of minimum energy-content.

Table 10.11 Design requirements for disposable cup

Function	Disposable hot-drink cup
Objectives	(a) Minimize energy-content and (b) Minimize cost, appropriately coupled
Constraints	(a) Stiff enough to be picked up (b) Thermally insulating

of environmental damage. Values $E^{\$}$ are, at present, unknown, but by taking extremes its influence can be explored.

The first term in this equation describes the material cost of the cup. It is the volume of material it contains (thought of as a cylinder of radius R, height h and wall thickness t, closed at one end) times the cost $C_m\rho$ per unit volume (C_m is the material cost per unit weight and ρ the density):

$$C = C_m m \approx (2\pi Rh + \pi R^2)tC_m\rho$$
$$= (2\alpha + 1)\pi R^2 t C_m\rho \quad (10.23)$$

where $\alpha = h/R$ the ratio of height-to-radius. The constraint on stiffness requires that ovalization must not become unacceptable when the cup is loaded across a diagonal, as in the figure. This imposes a limit on its stiffness, S:

$$S = \frac{F}{\delta} = \frac{C_1 EI}{R^3} = \frac{\alpha C_1 E t^3}{12R^2} > S_c \quad (10.24)$$

Here I is the second moment of area of the wall of the cup (proportional to $ht^3/12$ for a wall of uniform thickness, t), E is its Young's modulus, C_1 is a constant and S_c is the critical stiffness required for safe handling. Solving for t gives

$$t = \left(\frac{12R^2 S_c}{\alpha C_1 E}\right)^{1/3} \quad (10.25)$$

which, when inserted in equation (10.23), gives the *cost* of the cup:

$$C = C_m m = (2\alpha + 1)\pi R^2 C_m \rho \left(\frac{12R^2 S_c}{\alpha C_2 E} \right)^{1/3} \tag{10.26}$$

or

$$C = C_2 \left(\frac{C_m \rho}{E^{1/3}} \right) \tag{10.27}$$

in which the constant C_2 contains the design parameters. By a similar chain of argument, replacing $C_m \rho$ by $q\rho$ (where q is the energy per unit mass of the material), the *energy content* of the cup is

$$qm = C_2 \left(\frac{q\rho}{E^{1/3}} \right) \tag{10.28}$$

If we now associate a cost $E^{\$}$ with environmental impact as measured by energy content (an energy tax, for example, or a pollution tax), environmental impact can be converted to cost, giving:

$$V = C_2[M_1 + E^{\$} M_2] \tag{10.29}$$

with $M_1 = C_m \rho / E^{1/3}$ and $M_2 = q\rho / E^{1/3}$.

The selection: analytical method

Table 10.12 lists three candidates for the cup: foamed polystyrene (PS), polycarbonate (PC) and high density polyethylene (HDPE), with the relevant properties. The remaining columns list the wall thickness, the cost and the value, taking

$$R = 40\,\text{mm} \qquad \alpha = 4 \qquad C_1 = 24 \qquad S_c = 3\,\text{kN/m}$$

With no penalty on energy ($E^{\$} = 0$), polystyrene has the greatest value. A pollution tax of 0.01 \$/MJ leads to the ranking in the second last column; one of 0.05 \$/MJ gives the values in the last one. With the higher tax, PC becomes more attractive.

We have used numerical values for R, α, C_1 and S_c here, but it was not necessary. It is frequently so that the optimum selection is independent of some or all of the other variables of the design, and this is an example of just that. The variables R, α, C_1 and S_c are all contained in the quantity C_2 of equation (10.29), the value of which does not alter the ranking of the candidates in Table 10.12: ranking by V or by V/C_2.

Table 10.12 Value functions, V, for two values of exchange constant, $E^{\$}$

Material	ρ *Mg/m*^{3}	*E* *GPa*	C_m^* *\$/kg*	*q* *MJ/kg*	*t* *mm*	*C* *\$*	*V*, $E^{\$} =$ *−0.01 \$/MJ*	*V*, $E^{\$} =$ *−0.05 \$/MJ*
Expanded PS	0.05	0.03	1.4	180	2.7	0.009	**−0.02**	−0.07
Expanded PC	0.065	0.95	5.0	170	1	0.016	**−0.02**	**−0.04**
Expanded HDPE	0.08	0.006	1.6	150	4.6	0.3	−0.06	−0.17

*Cost of material in shape of cup, when mass produced, is almost the same as that of the material itself.

The selection: graphical method

Figure 10.10 shows M_1 plotted against M_2, allowing the selection of materials to minimize, in a balanced way, both cost and energy content. We will assume that the cups are at present pressed from solid polystyrene (PS) sheet with a density of $1060\,kg/m^3$; it is indicated as a black ellipse on the figure. The contours show the selection 'boundary' for various values of $E^\$$. The materials which lie below the appropriate contour are a better choice than the current material: they give a lower value of V than it does. For small values of $E^\$$, the contours are almost vertical; for large $E^\$$ they are almost horizontal.

Materials in the lower-left quadrant are both cheaper *and* less energy intensive than the current material: they are a better choice than the existing solid PS, regardless of the value of $E^\$$. Among these are a range of polyethylene foams, LDPE, with densities in the range 0.018 to $0.029\,Mg/m^3$ (LDPE (0.018), for instance) and the expanded polystyrenes with densities between 0.02 and $0.05\,Mg/m^3$ (EPS (0.02) or EPS (0.05)). But if the energy tax were high enough — if $E^\$$ were as high as 0.01 \$/MJ, for example — then a range of PVC foams become potential candidates; and if it rose to 0.1 \$/MJ, cups made of cork (!) would become economic. Table 10.13 summarizes the selection.

Further reading

Boustead, I. and Hancock, G.F. (1979) *Handbook of Industrial Energy Analysis*, Wiley, New York.

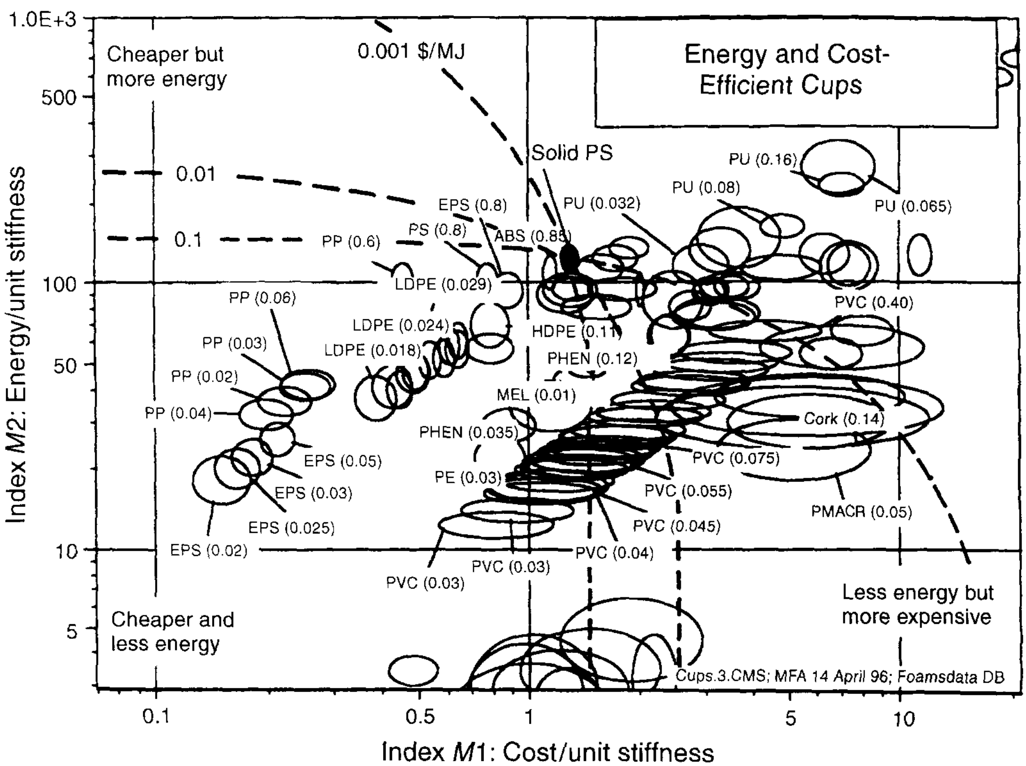

Fig. 10.10 Comparison of polystyrene with competing materials for disposable cups. (Figure created using *CMS* (1995) software.)

Table 10.13 Materials for low energy, cheap coffee cups

Material	*Comment*
Short design life	
Expanded polystyrene (EPS) foams [e.g. EPS(0.02) to EPS(0.05)]	The best choice: lower cost and energy content than solid PS; good thermal properties.
Polypropylene (PP) foams [e.g. PP(0.02) to PP(0.06)]	A viable alternative to expanded PS.
Polyethylene (LDPE) foams [e.g. LDPE(0.018) to LDPE(0.029)]	Considerably more expensive and more energy intensive than expanded PS.

Related case studies

Case Study 10.4: Compound objectives — materials for insulation

10.6 Summary and conclusions

Most designs are over-constrained: they must simultaneously meet several conflicting requirements. But although they conflict, an optimum selection is still possible. The 'active constraint' method, developed in Chapter 9, allows the selection of materials which optimally meet two or more constraints. It is illustrated here by two case studies, one of them mechanical, one electro-mechanical.

Greater problems arise when the design must meet two or more conflicting objectives (such as minimizing mass, cost and environmental impact). Here we need a way can be found to express all the objectives in the same units, a 'common currency', so to speak. The conversion factor is called the exchange constant, $E^{\$}$. Establishing the value of the exchange constant is an important step in solving the problem. With it, a value function V is constructed which combines the objectives. Materials which minimize V meet all the objectives in a properly balanced way. The most obvious common currency is cost itself, requiring an 'exchange rate' to be established between cost and the other objectives. This can be done for energy and for mass, and — at least in principle — for environmental impact. The method is illustrated by two further case studies.